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On sums of degrees of the partial quotients in continued fraction expansions of Laurent series

Mei-Ying Lü, Bao-Wei Wang, Jian Xu *

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, PR China

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ABSTRACT

For any formal Laurent series $x=\sum_{n=v}^{\infty} c_n z^{-n}$ with coefficients c_n lying in some given finite field, let $x=[a_0(x); a_1(x), a_2(x), \ldots]$ be its continued fraction expansion. It is known that, with respect to the Haar measure, almost surely, the sum of degrees of partial quotients $\deg a_1(x)+\cdots+\deg a_n(x)$ grows linearly. In this note, we quantify the exceptional sets of points with faster growth orders than linear ones by their Hausdorff dimension, which covers an earlier result by J. Wu.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of q elements and $\mathbb{F}_q((z^{-1}))$ denotes the field of all formal Laurent series with coefficients in $\mathbb{F}_q$. Recall that $\mathbb{F}_q[z]$ denotes the ring of polynomials in z with coefficients in $\mathbb{F}_q$.

For each $x=\sum_{n=v}^{\infty} c_n z^{-n} \in \mathbb{F}_q((z^{-1}))$, call $[x]=\sum_{n=v}^{0} c_n z^{-n} \in \mathbb{F}_q[z]$ the integer part of x and $\deg x=\inf\{n\in\mathbb{Z}\colon\ c_n\neq 0\}$ the degree of x, with the convention that if $x=0$, i.e. $c_n=0$ for all $n\in\mathbb{Z}$, put $\deg 0=-\infty$. Define the absolute value on $\mathbb{F}_q((z^{-1}))$ as

$$\|x\|=q^{\deg x}$$

which is a non-Archimedean absolute value. The field $\mathbb{F}_q((z^{-1}))$ is locally compact and complete under the metric $\rho(x,y)=\|x-y\|$.

Denote $I=\{x\in\mathbb{F}_q((z^{-1}))\colon\ \|x\|<1\}=\{x=\sum_{n=1}^{\infty} c_n z^{-n}\colon\ c_n\in\mathbb{F}_q\}$, which is the valuation ideal of $\mathbb{F}_q((z^{-1}))$. Let P be the Haar measure on $\mathbb{F}_q((z^{-1}))$ normalized to 1 on I.

Now we recall the regular continued fraction expansion over the field of formal Laurent series introduced by E. Artin [1]. Consider the transformation $T: I\to I$ defined by

$$Tx:=\frac{1}{x}-\left[\frac{1}{x}\right], \qquad T0:=0.$$

Then every $x\in I$ has the following continued fraction expansion:

$$x=\cfrac{1}{a_1(x)+\frac{1}{a_2(x)+\cdots}} := \left[0; a_1(x), a_2(x), \ldots\right],$$

* Corresponding author.
E-mail addresses: lmy19831102@hotmail.com (M.-Y. Lü), bwei_wang@yahoo.com.cn (B.-W. Wang), arielxj@hotmail.com (J. Xu).

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where the digits $a_n(x)$, called partial quotients, are polynomials of strictly positive degree and are defined by $a_n(x) = [\frac{1}{T^{n-1}(x)}]$ for all $n \geqslant 1$. Let $\{P_n(x)/Q_n(x),\ n \geqslant 0\}$ be the sequence of convergents in the expansion of x, i.e.

$$\frac{P_n(x)}{Q_n(x)} = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \cdots + \frac{1}{a_n(x)}}} := \bigl[0; a_1(x), a_2(x), \ldots, a_n(x)\bigr].$$

The metric and ergodic properties of this dynamical system have been studied extensively, see the works [2,6,7,13–17]. In particular, following from the ergodicity of T with respect to the Haar measure P, which was first proved by Niederreiter [13], one has

Theorem 1.1. *(See [2,13].) For P-almost all $x \in I$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \deg a_k(x) = \frac{q}{q-1}.$$

Theorem 1.1 implies that almost surely, the sum of degrees of the partial quotients grows linearly. In [18], J. Wu studied the cases with polynomial and exponential growth rate, i.e.

$$B(\beta,\gamma) = \left\{x \in I\colon \lim_{n\to\infty} \frac{1}{n^\beta} \sum_{k=1}^{n} \deg a_k(x) = \gamma \right\} \quad (\beta > 1,\ \gamma > 0),$$

$$C(\tau,\eta) = \left\{x \in I\colon \lim_{n\to\infty} \frac{1}{\tau^n} \sum_{k=1}^{n} \deg a_k(x) = \eta \right\} \quad (\tau > 1,\ \eta > 0)$$

by showing

Theorem 1.2. *(See [18].) For any $\beta > 1$, $\gamma > 0$, $\tau > 1$ and $\eta > 0$,*

(i) $\dim_{\mathrm{H}} B(\beta,\gamma) = \dfrac{1}{2}$,

(ii) $\dim_{\mathrm{H}} C(\tau,\eta) = \dfrac{1}{\tau+1}$,

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension.

In this paper, we would like to know what happens when polynomial orders γn^β $(\beta > 1,\ \gamma > 0)$ or exponential orders $\eta\tau^n$ $(\tau > 1,\ \eta > 0)$ are replaced by a general order faster than linear one.

Namely, we consider the size of the set

$$E(\phi) := \left\{x \in I\colon \lim_{n\to\infty} \frac{\deg a_1(x) + \cdots + \deg a_n(x)}{\phi(n)} = 1 \right\},$$

where $\phi : \mathbb{N} \to \mathbb{R}^+$ is a function satisfying $\phi(n)/n \to \infty$ as $n \to \infty$.

We prove

Theorem 1.3. *Let $\phi : \mathbb{N} \to \mathbb{R}^+$ satisfy $\phi(n)/n \to \infty$ as $n \to \infty$. Then either $E(\phi) = \emptyset$ or*

$$\dim_{\mathrm{H}} E(\phi) = \frac{1}{1+b}, \quad \text{with } b = \limsup_{n\to\infty} \frac{\phi(n+1)}{\phi(n)}.$$

To end this section, we outline the strategy in proving Theorem 1.3. The lower bound of $\dim_{\mathrm{H}} E(\phi)$ is quite easy, so the argument on upper bound constitutes the main substance of this note. In fact, we will show that $E(\phi)$ is a subset of Jarnik–Besicovitch set with some extra constraints. More precisely, for each $\alpha < b+1$ (see Lemma 2.5)

$$E(\phi) \subset E_\infty \cap J(\alpha),$$

where

$$E_\infty = \left\{x \in I\colon \lim_{n\to\infty} \frac{\deg a_1(x) + \cdots + \deg a_n(x)}{n} = \infty \right\}$$

and $J(\alpha)$ is the Jarnik–Besicovitch set over the field of formal Laurent series, i.e.

$$J(\alpha)=\left\{x\in I\colon \left\|x-\frac{P}{Q}\right\|<\frac{1}{\|Q\|^{\alpha}} \text{ for infinitely many } Q\in\mathbb{F}_q[z]\right\}.$$

We show

Theorem 1.4. *For any $\alpha\geqslant 2$,*

$$\dim_{\mathrm H}\bigl(E_\infty\cap J(\alpha)\bigr)=\dim_{\mathrm H}E_\infty\cdot\dim_{\mathrm H}J(\alpha)=\frac{1}{\alpha}.$$

There is an emerging topic on study diophantine approximation supported on fractals (see the works of D. Kleinbock, E. Lindenstrauss and B. Weiss [9,10], J. Levesley, C. Salp and S.L. Velani [12] or Y. Bugeaud [3]). So, Theorem 1.4 can also be viewed as an attempt on this topic.

2. Proofs of the results

We first collect some basic properties possessed by continued fractions of Laurent series, see H. Niederreiter [13], V. Berthé and H. Nakada [2] or M. Fuchs [6].

Lemma 2.1. *(See [6].) For any $x\in I$, let $P_n(x)/Q_n(x)$ denote the n-th convergent of x. Then we have*

(1) $1=\|Q_0(x)\|<\|Q_1(x)\|<\|Q_2(x)\|<\cdots$;
(2) $\|Q_n(x)\|=\prod_{k=1}^{n}\|a_k(x)\|=q^{\sum_{k=1}^{n}\deg a_k(x)}$;
(3) $\|x-P_n(x)/Q_n(x)\|=1/(\|Q_n(x)\|\cdot\|Q_{n+1}(x)\|)<1/\|Q_n(x)\|^2$.

Lemma 2.2. *(See [13].) For any $b_1,b_2,\dots,b_n\in\mathbb{F}_q[z]$ with strictly positive degree, call*

$$I(b_1,b_2,\dots,b_n)=\bigl\{x\in I\colon a_1(x)=b_1,\ a_2(x)=b_2,\ \dots,\ a_n(x)=b_n\bigr\}$$

an n-th order cylinder. Then the set $I(b_1,b_2,\dots,b_n)$ is a closed disc with center P_n/Q_n and diameter

$$\bigl|I(b_1,b_2,\dots,b_n)\bigr|=q^{-2\sum_{k=1}^{n}\deg b_k-1}.$$

Before proving Theorem 1.3, we give a remark on ϕ. For any given positive function $\phi(n)$, if $E(\phi)\neq\emptyset$, then there exists an $x_0\in E(\phi)$, define $\bar\phi(n)=\sum_{k=1}^{n}\deg a_k(x_0)$ for all $n\geqslant 1$. Obviously, we have $\phi(n)/\bar\phi(n)\to 1$ as $n\to\infty$, so $E(\phi)=E(\bar\phi)$. Hence, in what follows, we can always assume that $\phi:\mathbb{N}\to\mathbb{N}$ and $\phi(n+1)-\phi(n)\geqslant 1$ once $E(\phi)$ is non-empty.

2.1. *Lower bound of* $\dim_{\mathrm H}E(\phi)$

The lower bound is obtained by estimating the Hausdorff dimension of a homogeneous Moran subset of $E(\phi)$. We recall the definition and a basic dimensional result of the homogeneous Moran set at first, see [4,5,8] for details.

Let $\{n_k\}_{k\geqslant 1}$ be a sequence of positive integers and $\{c_k\}_{k\geqslant 1}$ be a sequence of positive numbers satisfying $n_k\geqslant 2$, $0<c_k<1$, $n_1c_1\leqslant\delta$ and $n_kc_k\leqslant 1$ $(k\geqslant 2)$, where δ is some positive number.

Let

$$D=\bigcup_{k\geqslant 0}D_k,\qquad D_0=\{\emptyset\},\qquad D_k=\bigl\{(i_1,\dots,i_k)\colon 1\leqslant i_j\leqslant n_j,\ 1\leqslant j\leqslant k\bigr\}.$$

If $\sigma=(\sigma_1,\dots,\sigma_k)\in D_k$, $\tau=(\tau_1,\dots,\tau_m)\in D_m$, we define the concatenation of σ and τ as

$$\sigma*\tau=(\sigma_1,\dots,\sigma_k,\tau_1,\dots,\tau_m).$$

Let (X,d) be a metric space. Suppose that $J\subset X$ is a closed subset with diameter $\delta>0$. A collection $\mathfrak{F}=\{J_\sigma\colon\ \sigma\in D\}$ of closed subsets of J is said to have a homogeneous Moran structure if it satisfies:

(1) $J_\emptyset=J$;
(2) For any $k\geqslant 1$ and $\sigma\in D_{k-1}$, $J_{\sigma*1},J_{\sigma*2},\dots,J_{\sigma*n_k}$ are subsets of J_σ and $\mathrm{int}(J_{\sigma*i})\cap\mathrm{int}(J_{\sigma*j})=\emptyset$ $(i\neq j)$, where $\mathrm{int}\,A$ denotes the interior of A;
(3) For any $k\geqslant 1$ and $\sigma\in D_{k-1}$, $1\leqslant j\leqslant n_k$, we have

$$\frac{|J_{\sigma*j}|}{|J_\sigma|}=c_k.$$

If $\mathfrak{F}$ is such a collection, $E:=\bigcap_{k\geqslant 1}\bigcup_{\sigma\in D_k}J_\sigma$ is called a homogeneous Moran set determined by $\mathfrak{F}$.

Lemma 2.3. *(See [5,8].) For the above defined homogeneous Moran set, we have*

$$\dim_{\mathrm{H}} E \geqslant \liminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_{k+1} n_{k+1}}.$$

Lemma 2.4. *Assume that $\{s_n\}_{n=1}^{\infty}$ is a sequence of positive integers satisfying $\frac{1}{n}\sum_{k=1}^{n} s_k \to \infty$ as $n \to \infty$. Let $E = \{x \in I: \deg a_n(x) = s_n\}$. Then we have*

$$\dim_{\mathrm{H}} E \geqslant \liminf_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{2(s_1 + s_2 + \cdots + s_n) + s_{n+1}}.$$

Proof. Let

$$D_n = \{\sigma\colon \sigma = (a_1, a_2, \ldots, a_n) \in \mathbb{F}_q[z]^n,\ \deg a_k = s_k,\ 1 \leqslant k \leqslant n\}.$$

Put

$$E_0 = I, \qquad E_n = \bigcup_{(a_1,a_2,\ldots,a_n)\in D_n} I(a_1, a_2, \ldots, a_n), \quad \forall n \geqslant 1.$$

Then

$$E = \bigcap_{n=1}^{+\infty} E_n.$$

Take $n_k = (q-1)q^{s_k}$, $c_k = q^{-2s_k}$. From the above structure, it follows that each component $I(a_1, a_2, \ldots, a_{k-1})$ in E_{k-1} contains n_k many elements $I(a_1, a_2, \ldots, a_k)$ in E_k with the same ratio c_k. Thus E is a standard homogeneous Moran set. By Lemma 2.3, we have

$$\begin{aligned}\dim_{\mathrm{H}} E &\geqslant \liminf_{n\to\infty} \frac{n\log(q-1) + (s_1 + \cdots + s_n)\log q}{-\log(q-1) + (2(s_1 + \cdots + s_n) + s_{n+1})\log q}\\ &= \liminf_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{2(s_1 + s_2 + \cdots + s_n) + s_{n+1}}. \qquad \square\end{aligned}$$

Take $s_1 = \phi(1)$, $s_n = \phi(n) - \phi(n-1)$ for $n \geqslant 2$. It is easy to see that $E \subset E(\phi)$. Lemma 2.4 gives

$$\dim_{\mathrm{H}} E(\phi) \geqslant \liminf_{n\to\infty} \frac{\phi(n)}{\phi(n) + \phi(n+1)} = \frac{1}{1+b}.$$

So, we get the desired lower bound of the Hausdorff dimension of $E(\phi)$.

2.2. Upper bound of $\dim_{\mathrm{H}} E(\phi)$

The estimation of the upper bound constitutes the main substance of the proof. Since ϕ is monotonic increasing, we have $b \geqslant 1$.

Lemma 2.5. *For any $\alpha < b+1$,*

$$E(\phi) \subset E_\infty \cap J(\alpha).$$

Proof. It is evident that $E(\phi) \subset E_\infty$ since $\phi(n)/n \to \infty$ as $n \to \infty$.

For any $x \in E(\phi)$, one can see that

$$\limsup_{n\to\infty} \frac{\deg a_1(x) + \cdots + \deg a_n(x) + \deg a_{n+1}(x)}{\deg a_1(x) + \cdots + \deg a_n(x)} = \limsup_{n\to\infty} \frac{\phi(n+1)}{\phi(n)} = b.$$

Since $\deg Q_n(x) = \sum_{k=1}^{n} \deg a_k(x)$ (see Lemma 2.1(2)), above formula is nothing but

$$\limsup_{n\to\infty} \frac{\log_q \|Q_{n+1}(x)\|}{\log_q \|Q_n(x)\|} = b.$$

By Lemma 2.1(3), we know that

$$\left\| x - \frac{P_n(x)}{Q_n(x)} \right\| = \frac{1}{\|Q_n(x)Q_{n+1}(x)\|},$$

thus, for any $\alpha < b+1$,

$$\left\| x - \frac{P_n(x)}{Q_n(x)} \right\| < \frac{1}{\|Q_n(x)\|^{\alpha}},$$

holds for infinitely many $n \in \mathbb{N}$. $\square$

Remark 2.6. Note that, when $\alpha \leqslant 2$,

$$J(\alpha) = \{x \in I\colon \text{ the continued fraction expansion of } x \text{ is infinite}\}.$$

So, Lemma 2.5 still holds if we choose

$$\begin{cases} \alpha = 2, & \text{if } b = 1, \\ 2 < \alpha < b+1, & \text{if } b > 1. \end{cases}$$

In view of Lemma 2.5, $\dim_{\mathrm{H}} E_\infty \cap J(\alpha)$ will give a upper bound for $\dim_{\mathrm{H}} E(\phi)$. Hence, we are led to prove Theorem 1.4.

Proof of Theorem 1.4.

(I) $\dim_{\mathrm{H}}(E_\infty \cap J(\alpha)) \geqslant 1/\alpha$.

We will apply Lemma 2.4 to a subset of $E_\infty \cap J(\alpha)$ to give the lower bound estimation on $\dim_{\mathrm{H}}(E_\infty \cap J(\alpha))$.

Define an integer sequence $\{s_n\}_{n \geqslant 1}$ recursively as

$$s_n = \begin{cases} [(n+1)^{\frac{3}{2}}] + 1, & \text{when } n \neq k^2, \\ [(\alpha - 2)\sum_{j=1}^{n-1} s_j] + 1, & \text{when } n = k^2. \end{cases}$$

Then take

$$F = \bigl\{x \in I\colon \deg a_n(x) = s_n, \text{ for all } n \geqslant 1\bigr\}.$$

We claim that $F \subset E_\infty \cap J(\alpha)$.

(1) $F \subset E_\infty$. Whether $\alpha = 2$ or not, for any $x \in F$,

$$\sum_{i=1}^{n} \deg a_i(x) \geqslant \max\{s_{n-1}, s_n\} \geqslant n^{\frac{3}{2}}.$$

(2) $F \subset J(\alpha)$. When $\alpha = 2$, the inclusion is trivial (see Remark 2.6).
When $\alpha > 2$, for any $x \in F$, by Lemma 2.1(2), for each $k \geqslant 2$,

$$\deg Q_{k^2}(x) = \sum_{i=1}^{k^2} \deg a_i(x) > (\alpha - 1)\sum_{i=1}^{k^2-1} \deg a_i(x) = (\alpha - 1)\deg Q_{k^2-1}(x).$$

As a result, for each $k \geqslant 2$,

$$\left\| x - \frac{P_{k^2-1}(x)}{Q_{k^2-1}(x)} \right\| = \frac{1}{\|Q_{k^2-1}(x) Q_{k^2}(x)\|} < \frac{1}{\|Q_{k^2-1}(x)\|^{\alpha}}.$$

Then an application of Lemma 2.4 to the set F gives the desired result.

(II) $\dim_{\mathrm{H}}(E_\infty \cap J(\alpha)) \leqslant 1/\alpha$.

We begin with the construction of a family of measures $\{\mu_t,\ t > 1\}$. For any $t > 1$ and $\{b_1, b_2, \ldots, b_n\} \subset \mathbb{F}_q[z]$ with $\deg b_j \geqslant 1$ $(1 \leqslant j \leqslant n)$, set

$$\mu_t\bigl(I(b_1, b_2, \ldots, b_n)\bigr) = q^{-t\sum_{j=1}^{n} \deg b_j - nP(t)},$$

where

$$P(t) = \log_q\bigl(q(q-1)\bigr) - \log_q\bigl(q^t - q\bigr).$$

It is easy to check that

$$\sum_{b_{n+1}} \mu_t\bigl(I(b_1, b_2, \ldots, b_{n+1})\bigr) = \mu_t\bigl(I(b_1, b_2, \ldots, b_n)\bigr),$$

$$\sum_{b_1, b_2, \ldots, b_n} \mu_t\bigl(I(b_1, b_2, \ldots, b_n)\bigr) = 1,$$

where the sum is taken over all b_j such that $\deg b_j \geqslant 1$. So the measure μ_t is well defined.

Fix $t > 1$ and $\epsilon > 0$. Take $M_0(t,\epsilon) \in \mathbb{N}$ such that for all $M \geqslant M_0(t,\epsilon)$,

$$M > \frac{P(t)}{\epsilon}. \tag{2.1}$$

Now we give a cover of the set $E_\infty \cap J(\alpha)$. For any $n \geqslant 1$, let

$$I_n(M) = \left\{(a_1,\dots,a_n) \in \mathbb{F}_q[z]^n\colon \sum_{j=1}^{n} \deg a_j > nM\right\}. \tag{2.2}$$

Clearly, $E_\infty \cap J(\alpha)$ is contained in

$$\left\{x \in I\colon \big(a_1(x),\dots,a_n(x)\big) \in I_n(M) \text{ and } \deg a_{n+1}(x) > (\alpha-2)\sum_{j=1}^{n} \deg a_j(x) \text{ simultaneously for infinitely many } n\right\}.$$

For any $(a_1,\dots,a_n) \in I_n(M)$, set

$$J(a_1,\dots,a_n) = \bigcup_{a_{n+1}\colon \deg a_{n+1} > (\alpha-2)\sum_{j=1}^{n}\deg a_j} I(a_1,\dots,a_n,a_{n+1}).$$

Then

$$E_\infty \cap J(\alpha) \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{(a_1,\dots,a_n)\in I_n(M)} J(a_1,\dots,a_n). \tag{2.3}$$

We will estimate the diameter of $J(a_1,\dots,a_n)$ for all $(a_1,\dots,a_n) \in I_n(M)$. We have

$$\big|J(a_1,\dots,a_n)\big| \leqslant \sum_{a_{n+1}\colon \deg a_{n+1} > (\alpha-2)\sum_{j=1}^{n}\deg a_j} \big|I(a_1,\dots,a_n,a_{n+1})\big| \ll \big|I(a_1,\dots,a_n)\big|^{\frac{\alpha}{2}},$$

where the implied constant in the Vinogradov symbol ($\ll$) is absolute. Subsequently, for any $(a_1,\dots,a_n) \in I_n(M)$,

$$\big|I(a_1,\dots,a_n)\big|^{\frac{t+\varepsilon}{2}} \leqslant q^{-(t+\varepsilon)\sum_{j=1}^{n}\deg a_j} \leqslant q^{-t\sum_{j=1}^{n}\deg a_j - nP(t)} = \mu_t\big(I(a_1,\dots,a_n)\big),$$

where the second inequality follows from $\sum_{i=1}^{n}\deg a_i \geqslant nM$ (formula 2.2) and $M \geqslant P(t)/\epsilon$ (formula 2.1). After these preliminaries, we see that the $\frac{t+\epsilon}{\alpha}$-dimensional Hausdorff measure

$$\begin{aligned}
\mathcal{H}^{\frac{t+\varepsilon}{\alpha}}\big(E_\infty \cap J(\alpha)\big) &\leqslant \limsup_{n\to\infty} \sum_{(a_1,\dots,a_n)\in I_n(M)} \big|J(a_1,\dots,a_n)\big|^{\frac{t+\varepsilon}{\alpha}} \\
&\ll \limsup_{n\to\infty} \sum_{(a_1,\dots,a_n)\in I_n(M)} \Big[\big|I(a_1,\dots,a_n)\big|^{\frac{\alpha}{2}}\Big]^{\frac{t+\varepsilon}{\alpha}} \\
&\ll \limsup_{n\to\infty} \sum_{(a_1,\dots,a_n)\in I_n(M)} \mu_t\big(I(a_1,\dots,a_n)\big) \ll 1.
\end{aligned}$$

So, we have

$$\dim_{\mathrm{H}}\big(E_\infty \cap J(\alpha)\big) \leqslant \frac{t+\varepsilon}{\alpha}.$$

Letting $\varepsilon \to 0$ and $t \to 1$, we obtain

$$\dim_{\mathrm{H}}\big(E_\infty \cap J(\alpha)\big) \leqslant \frac{1}{\alpha}.$$

Thus, combine (I) and (II), we arrive at

$$\dim_{\mathrm{H}}\big(E_\infty \cap J(\alpha)\big) = \frac{1}{\alpha}. \tag{2.4}$$

Take $\alpha = 2$. From Remark 2.6, we have $E_\infty \cap J(\alpha) = E_\infty$. So

Corollary 2.7. $\dim_{\mathrm{H}} E_\infty = 1/2$.

It is well known that $\dim_{\mathrm{H}} J(\alpha) = 2/\alpha$ (see S. Kristensen [11]), so it follows that

$$\dim_{\mathrm{H}}\big(E_\infty \cap J(\alpha)\big) = \dim_{\mathrm{H}} E_\infty \cdot \dim_{\mathrm{H}} J(\alpha).$$

This complete the proof of Theorem 1.4. □

Proof of Theorem 1.3. Lemma 2.5 together with Theorem 1.4 immediately follows that

$$\dim_{\mathrm{H}} E(\phi) \leqslant \frac{1}{1+b},$$

as was to be proved. □

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